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STABILIZATION OF A NATURAL MECHANICAL SYSTEM WITHOUT MEASURING ITS VELOCITIES WITH APPLICATION TO THE CONTROL OF A RIGID BODY†

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A method for the asymptotic stabilization of a natural mechanical system is proposed which does not require measurements of the velocities of the system, but requires the solution of linear differential equations during the control process. © 1999 Elsevier Science Ltd. All rights reserved.

Various methods have been proposed to solve the problem of the asymptotic stabilization of a mechanical system by external control forces; most of them require measurements of velocities. Below we propose control laws that require measurements only of coordinates rather than velocities; in the process, use is made of observers (filters). It should be noted that coordinate detectors are less "noisy" than velocity detectors; in addition, observers are cheaper to install than velocity detectors. The main advantage of the stabilization schemes proposed here is their practical simplicity compared with previously proposed stabilization methods for the programmed motion of robot-manipulators and controlled satellites.

Some results of this paper were presented in my candidate dissertation.‡

1. THE EQUATIONS OF DYNAMICS

The dynamics of a natural mechanical system are described by a Lagrange equation of the second kind

$$(\partial K(\mathbf{q},\dot{\mathbf{q}})/\partial\ddot{\mathbf{q}})-\partial K(\mathbf{q},\dot{\mathbf{q}})/\partial\mathbf{q}+\mathbf{g}(\mathbf{q})=\mathbf{M} \tag{1.1}$$

which is equivalent to the equation

$$A(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{g}(\mathbf{q})=\mathbf{M} \tag{1.2}$$

where $\mathbf{q}\in R^n$ is a vector of generalized coordinates, $A(\mathbf{q})$ is a positive-definite symmetric inertia matrix, $K=\dot{\mathbf{q}}^*A(\mathbf{q})\dot{\mathbf{q}}/2$ is the kinetic energy, $\mathbf{g}(\mathbf{q})$ is a vector of potential forces, $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$ is a vector of terms quadratic in $\dot{\mathbf{q}}$ and $\mathbf{M}$ is a vector of generalized control forces.

Let $\mathbf{q}_p(t)\in C^2[0,\infty)$ be the programmed motion vector and let the functions $\dot{\mathbf{q}}_p(t)$, $\ddot{\mathbf{q}}_p(t)$ be bounded on the semiaxis. We will assume that the matrices $A(\mathbf{q})$ and $A^{-1}(\mathbf{q})$, the vectors $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{g}(\mathbf{q})$ and their first and second partial derivatives with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$ are uniformly bounded in some neighbourhood of the programmed motion. We will also assume that $|\dot{\mathbf{q}}^*A(\mathbf{q})\dot{\mathbf{q}}|>c_1|\dot{\mathbf{q}}|^2$, $|\dot{\mathbf{q}}^*A^{-1}(\mathbf{q})\dot{\mathbf{q}}|>c_{-1}|\dot{\mathbf{q}}|^2$, $\forall\,\mathbf{q}\in R^n$, $c_1, c_{-1}=\text{const}>0$ where the asterisk denotes transposition and $|\cdot|$ is the Euclidean vector norm.

The programmed generalized force is defined by the formula

$$\mathbf{M}_p(t)=A(\mathbf{q}_p)\ddot{\mathbf{q}}_p+\mathbf{b}(\mathbf{q}_p,\dot{\mathbf{q}}_p)+\mathbf{g}(\mathbf{q}_p) \tag{1.3}$$

The vector-valued function of time $\mathbf{M}_p(t)$ is a solution of the inverse problem of dynamics for the appropriate equations. This function may be computed in advance and stored, e.g. as a spline.

We will introduce the deviation variables $\mathbf{x}=\mathbf{q}-\mathbf{q}_p$, $\mathbf{y}=\dot{\mathbf{q}}-\dot{\mathbf{q}}_p$.

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2. ASYMPTOTIC STABILIZATION OF THE PROGRAMMED POSITION

Suppose the desired motion is a constant vector $\mathbf{q}_p(t)$ = const. Consider the regulator

$$\mathbf{M} = -K_1(\mathbf{q}-\mathbf{q}_p) - K_2(\mathbf{q}-\hat{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) \tag{2.1}$$

and the auxiliary differential equation

$$\dot{\hat{\mathbf{q}}} = K_3(\mathbf{q}-\hat{\mathbf{q}}) \tag{2.2}$$

which is equivalent to the equation

$$\dot{\mathbf{w}} = -K_3\mathbf{w} + \dot{\mathbf{q}} \quad (\mathbf{w} = \mathbf{q}-\hat{\mathbf{q}}) \tag{2.3}$$

where K_1 is a positive-definite symmetric matrix and K_2 and K_3 are diagonal matrices with positive diagonal elements.

Theorem 2.1. The closed-loop system (1.1), (2.1), (2.2) ($\mathbf{q}-\mathbf{q}_p$, $\dot{\mathbf{q}}$, $\mathbf{q}-\hat{\mathbf{q}}$) is, as a whole, asymptotically stable.

Proof. Without loss of generality, we may assume that $\mathbf{q}_p = 0$ (otherwise, we introduce the change of variables $\mathbf{x} = \mathbf{q}-\mathbf{q}_p$). Introducing the generalized momentum $\mathbf{p} = A(\mathbf{q})\dot{\mathbf{q}}$, we can write the equations of motion in Hamiltonian form

$$\dot{\mathbf{q}} = \frac{\partial H(\mathbf{p},\mathbf{q})}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H(\mathbf{p},\mathbf{q})}{\partial \mathbf{q}} - K_1\mathbf{q} - K_2\mathbf{w}$$

where $H(\mathbf{p}, \mathbf{q})$ is the kinetic energy of the system in terms of $\mathbf{q}$ and $\mathbf{p}$.

Consider the Lyapunov function

$$V = H(\mathbf{p}, \mathbf{q}) + \mathbf{q}^*K_1\mathbf{q}/2 + \mathbf{w}^*K_2\mathbf{w}/2$$

Its derivative along trajectories of system (1.1), (2.1), (2.2) is

$$\begin{aligned}\dot{V} &= \frac{\partial H^*}{\partial \mathbf{q}}\dot{\mathbf{q}} + \mathbf{q}^*K_1\dot{\mathbf{q}} + \frac{\partial H^*}{\partial \mathbf{p}}\dot{\mathbf{p}} + \mathbf{w}^*K_2\dot{\mathbf{w}} = \\ &= \frac{\partial H^*}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} + \mathbf{q}^*K_1\frac{\partial H}{\partial \mathbf{p}} + \frac{\partial H^*}{\partial \mathbf{p}}\left(-\frac{\partial H}{\partial \mathbf{q}} - K_1\mathbf{q} - K_2\mathbf{w}\right) + \mathbf{w}^*K_2(-K_3\mathbf{w} + \dot{\mathbf{q}}) = \\ &= -\dot{\mathbf{q}}^*K_2\mathbf{w} - \mathbf{w}^*K_2K_3\mathbf{w} + \mathbf{w}^*K_2\dot{\mathbf{q}} = -\mathbf{w}^*K_2K_3\mathbf{w}.\end{aligned}$$

Let us investigate the set $S = \{\mathbf{q}, \dot{\mathbf{q}}, \mathbf{w}\colon \dot{V} = 0\}$. Using the condition $\mathbf{w} = 0$ and Eq. (2.3), we obtain $\dot{\mathbf{q}} = 0$. It follows from the second equation in the Hamiltonian system or the equations of dynamics and the regulator (1.1), (2.1), with the conditions $\mathbf{w} = 0$ and $\dot{\mathbf{q}} = 0$, that $\mathbf{q} = 0$. Thus, the set S consists of the single point (0, 0, 0) which, by the Barbashin–Krasovskii theorem, is asymptotically stable as a whole.

In practice, the real-time computation of $\mathbf{g}(\mathbf{q})$ may involve difficulties. To overcome this obstacle, one can use the regulator

$$\mathbf{M} = -K_1(\mathbf{q}-\mathbf{q}_p) - K_2(\mathbf{q}-\hat{\mathbf{q}}) + \mathbf{g}(\mathbf{q}_p) \tag{2.4}$$

Theorem 2.2. The closed-loop system (1 1), (2.2), (2.4) ($\mathbf{q}-\mathbf{q}_p$, $\dot{\mathbf{q}}$, $\mathbf{q}-\hat{\mathbf{q}}$) is asymptotically stable, provided the matrix $K_1 + g_q(\mathbf{q}_p)$ is positive definite and the matrices K_2 and K_3 are diagonal with positive diagonal elements.

Proof. The closed-loop system is governed by the equation

$$A(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) = -K_1(\mathbf{q}-\mathbf{q}_p) - K_2(\mathbf{q}-\hat{\mathbf{q}}) - (\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_p))$$

and by Eq. (2.2). The linear approximation of this system is described by the following equations in the variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$

$$\dot{\mathbf{x}} = \mathbf{y}, \quad \ddot{\mathbf{x}} = A^{-1}(\mathbf{q}_p)[-(K_1 + g_q(\mathbf{q}_p))\mathbf{x} - K_2\mathbf{w}], \quad \dot{\mathbf{w}} = -K_3\mathbf{w} + \dot{\mathbf{q}}$$

The subscript q indicates partial differentiation.

By Theorem 2.1, this system is asymptotically stable. Consequently, its eigenvalues lie in the open left half-plane, and so system (1.1), (2.2), (2.4) is asymptotically stable.

3. STABILIZATION OF THE PROGRAMMED PATH

Consider the regulator

$$\mathbf{M} = \mathbf{M}_p(t) - \gamma(K_0\mathbf{x} + \varkappa K_1\mathbf{w}) \quad (\mathbf{w} = \upsilon K_4\mathbf{q} - \hat{\mathbf{q}}) \tag{3.1}$$

where the vector $\hat{\mathbf{q}}$ is an estimate for the vector $\upsilon K_4\mathbf{q}$. The control process entails solving the linear equation

$$\dot{\hat{\mathbf{q}}} = \upsilon K_3(\upsilon K_4\mathbf{q} - \hat{\mathbf{q}}) + \upsilon K_4\dot{\mathbf{q}}_p \tag{3.2}$$

where K_i are diagonal matrices with positive diagonal elements, and $\upsilon, \kappa, \gamma > 0$.

Adding the vector $-\upsilon K_4\dot{\mathbf{q}}$ to both sides of Eq. (3.2), we obtain

$$\dot{\mathbf{w}} = -\upsilon K_3\mathbf{w} + \upsilon K_4\mathbf{y} \tag{3.3}$$

Theorem 3.1. The closed-loop system (1.1), (3.1), (3.2) is uniformly (with respect to time) asymptotically stable relative to the variables **x**, **y**, **w** for all sufficiently large $\gamma, \kappa, \upsilon > 0$.

Proof. System (1.1), (3.1), (3.2) may be written in the form

$$\begin{gathered}\dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{y}} = -\ddot{\mathbf{q}}_p + A^{-1}(\mathbf{x} + \mathbf{q}_p)[-\gamma K_0\mathbf{x} - \gamma\varkappa K_1\mathbf{w} + \mathbf{M}_p - \mathbf{B}(\mathbf{x} + \mathbf{q}_p, \mathbf{y} + \dot{\mathbf{q}}_p)] \\ \dot{\mathbf{w}} = -\upsilon K_3\mathbf{w} + \upsilon K_4\mathbf{y} \quad (\mathbf{B}(\mathbf{x} + \mathbf{q}_p, \mathbf{y} + \dot{\mathbf{q}}_p) = \mathbf{b}(\mathbf{x} + \mathbf{q}_p, \mathbf{y} + \dot{\mathbf{q}}_p) + \mathbf{g}(\mathbf{x} + \mathbf{q}_p))\end{gathered} \tag{3.4}$$

Expand the right-hand sides of the first two equations in Taylor series in powers of **x**, **y**. After reduction we obtain

$$\begin{gathered}\dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{y}} = (-A^{-1}(\mathbf{q}_p)\gamma K_0 + L(t))x - A^{-1}(\mathbf{q}_p)\gamma\varkappa K_1\mathbf{w} + N(t)\mathbf{y} + \ldots \\ (L(t) = -A^{-1}(\mathbf{q}_p)\mathbf{B}_q(\mathbf{q}_p, \dot{\mathbf{q}}_p) + A_q^{-1}(\mathbf{q}_p)A(\mathbf{q}_p)\ddot{\mathbf{q}}_p, \quad N(t) = -A^{-1}(\mathbf{q}_p)B_{\dot{q}}(\mathbf{q}_p, \dot{\mathbf{q}}_p))\end{gathered} \tag{3.5}$$

where the dots represent non-linear terms.

We will use Klimushev's theorem [1]. The degenerate system of the linear approximation obtained by putting $\upsilon = \infty$, that is, $1/\upsilon = 0$, is equivalent to system (3.5) with $K_1\mathbf{w}$ replaced by $K_5\mathbf{y}$. The truncated linear approximation system has the form $\mathbf{w} = -K_3\mathbf{w}$ and is obviously uniformly asymptotically stable.

We will show that the degenerate system is uniformly asymptotically stable. Once again, we apply Klimushev's theorem [1], assuming that the parameter $1/\gamma$ is small. The corresponding degenerate system of the linear approximation has the following form when $\gamma = \infty$

$$\dot{\mathbf{x}} = \mathbf{y}, \quad 0 = -A^{-1}(\mathbf{q}_p(t))K_0\mathbf{x} - A^{-1}(\mathbf{q}_p(t))\varkappa K_5\mathbf{y}$$

and is equivalent to the asymptotically stable equation

$$\dot{\mathbf{x}} = -\varkappa^{-1}K_5^{-1}K_0\mathbf{x}$$

To prove that the truncated system

$$\dot{\mathbf{y}} = -\varkappa A^{-1}(\mathbf{q}_p(t))K_5\mathbf{y} \tag{3.6}$$

is uniformly asymptotically stable, we consider the Lyapunov function

$$V = \mathbf{y}^* A(\mathbf{q}_p(t))\mathbf{y}/2$$

whose derivative along trajectories of system (3.6) is

$$\dot{V} = -\varkappa y^* K_5 \mathbf{y} + \mathbf{y}^* \dot{A}(\dot{\mathbf{q}}_p(t))\mathbf{y}/2$$

Under our assumptions, $\dot{A}(\dot{\mathbf{q}}_p(t))$ is a bounded matrix-valued function of time. For all sufficiently large κ, the function $\dot{V}$ will be negative definite, and the functions V and $\dot{V}$ will satisfy estimates characteristic for quadratic forms [2], implying that the truncated system is indeed uniformly asymptotically stable. This completes the proof.

The theorem may also be proved using a theorem due to Hoppensteadt [3] rather than Klimushev's theorem.

We will now consider a regulator that does not require the evaluation of $\mathbf{M}_p(t)$ and so does not entail knowing the exact form of the equations of dynamics

$$\mathbf{M} = -\gamma(K_0\mathbf{x} + K_1\mathbf{w}) \tag{3.7}$$

Theorem 3.2. For any ε-tube (ε > 0) of the programmed path $(\mathbf{q}_p(t), \dot{\mathbf{q}}_p(t))$, the motion of system (1.1), (3.2), (3.7) will occur within that tube for all $t \geqslant 0$ and all sufficiently large $\gamma > 0$, provided that the initial mismatch $|\mathbf{x}(0)| + |\mathbf{y}(0)| + |\mathbf{w}(0)|$ is sufficiently small. If the initial mismatch is not small, then for any $t_0 > 0$ there exist $\upsilon^*, \gamma^* > 0$ such that, for all $\gamma > \gamma^*$, $\upsilon > \upsilon^*$, the motion will occur within the ε-tube for all $t \geqslant t_0$.

The proof is carried out using singular perturbation theory [4, 5], first with the small parameter $1/\upsilon$ and then with the small parameter $1/\gamma$.

Remarks. 1. The result remains true if the left-hand side of Eq. (1.1) includes bounded interference.

2. If $1/\gamma$ and $1/\upsilon$ are decreased, the distance between the solutions of the degenerate and the initial systems decreases (linearly) at the same rate. It follows that the control **M** is bounded as γ and υ increase (provided that the initial mismatch is small enough).

Although the proofs of the theorems in [1, 3] are based on the application of Lyapunov functions, it is quite difficult to derive from those proofs an estimate of the order of magnitude of the parameter, i.e. of the gain.

At large mismatches, the required control becomes too large and is physically impracticable. If the mismatch becomes very large at some time t_x, it is recommended that a new programmed path $\mathbf{q}_{pn}(t)$ should be constructed for which $\mathbf{q}_{pn}(t_x) = \mathbf{q}(t_x)$, $\dot{\mathbf{q}}_{pn}(t_x) = \dot{\mathbf{q}}(t_x)$ and which, beginning at a certain time t_y $(t_y > t_x)$, must coincide with the old path.

If the parameters of the system are known, it is preferable to use a regulator based on solving the inverse problem of dynamics. When one is using the control scheme (3.2), (3.7), the rotation of the motors may switch direction frequently in the neighbourhood of the programmed motion.

4. STABILIZATION OF THE PROGRAMMED POSITION OF A PLANE ELASTIC MANIPULATOR

The dynamics of a plane manipulator with elastic hinges when there is no gravity force, may be described by the following equations [6, 7]

$$(\partial K/\partial \dot{\mathbf{q}}_1)^{\cdot} - \partial K/\partial \mathbf{q}_1 + k(\mathbf{q}_1 - \mathbf{q}_2) = 0, \quad J\ddot{\mathbf{q}}_2 - k(\mathbf{q}_1 - \mathbf{q}_2) = \mathbf{M} \tag{4.1}$$

where $\mathbf{q}_1 \in R^n$ is the vector of angular coordinates of the links, $\mathbf{q}_2 \in R^n$ is the vector of angular coordinates of the motor rotors, $K = \dot{\mathbf{q}}_1^* A(\mathbf{q}_1)\dot{\mathbf{q}}_1/2$ is the kinetic energy of the links, $A(\mathbf{q}_1)$ is the positive definite inertia matrix of the links, J is the diagonal inertia matrix of the motor rotors (with positive diagonal elements), k is the diagonal stiffness matrix of the hinges (with positive diagonal elements) and $\mathbf{M} \in R_n$ is the vector of control torques applied to the rotor.

Let $\mathbf{q}_{1p} = \mathbf{q}_{2p} =$ const be the desired position. The equality $\mathbf{q}_{1p} = \mathbf{q}_{2p}$ corresponds to the unstressed state of the manipulator hinges.

Consider the regulator

$$\mathbf{M} = -K_0(\mathbf{q}_2 - \mathbf{q}_{2p}) - K_2(\mathbf{q}_2 - \hat{\mathbf{q}}_2) \tag{4.2}$$

and the auxiliary equation

$$\dot{\hat{\mathbf{q}}}_2 = K_3(\mathbf{q}_2 - \hat{\mathbf{q}}_2) \tag{4.3}$$

which may be solved during the control process. The last equation is equivalent to

$$\dot{\mathbf{w}}_2 = -K_3\mathbf{w}_2 + \dot{\mathbf{q}}_2 \quad (\mathbf{w}_2 = \mathbf{q}_2 - \hat{\mathbf{q}}_2) \tag{4.4}$$

where K_i are diagonal matrices with positive diagonal elements.

Theorem 4.1. The closed-loop system (4.1), (4.2), (4.3) ($\mathbf{q}_1 - \mathbf{q}_{1p}$, $\dot{\mathbf{q}}_1$, $\mathbf{q}_2 - \mathbf{q}_{2p}$, $\dot{\mathbf{q}}_2$, $\mathbf{q}_2 - \hat{\mathbf{q}}_2$) is asymptotically stable on the whole.

Proof. Using the momentum vector $\mathbf{p}_1 = A(\mathbf{q}_1)\dot{\mathbf{q}}_1$, we write the equations of dynamics as follows:

$$\dot{\mathbf{q}}_1 = \frac{\partial H(\mathbf{p}_1,\mathbf{q}_1)}{\partial \mathbf{p}_1}, \quad \dot{\mathbf{p}}_1 = -\frac{\partial H(\mathbf{p}_1,\mathbf{q}_1)}{\partial \mathbf{q}_1} - k(\mathbf{q}_1 - \mathbf{q}_2), \quad \ddot{\mathbf{q}}_2 = J^{-1}(\mathbf{M} + k(\mathbf{q}_1 - \mathbf{q}_2)) \tag{4.5}$$

where $H(\mathbf{p}_1, \mathbf{q}_1)$ is the kinetic energy of the links in terms of the variables $\mathbf{p}_1$ and $\mathbf{q}_1$.

We consider a Lyapunov function of the "kinetic energy plus quadratic form" type

$$V(\mathbf{p}_1,\mathbf{q}_1,\dot{\mathbf{q}}_2) = H(\mathbf{p}_1,\mathbf{q}_1) + [\dot{\mathbf{q}}_2^* J\dot{\mathbf{q}}_2 + (\mathbf{q}_1 - \mathbf{q}_2)^* k(\mathbf{q}_1 - \mathbf{q}_2) +$$
$$+(\mathbf{q}_2 - \mathbf{q}_{2p})^* K_0(\mathbf{q}_2 - \mathbf{q}_{2p}) + \mathbf{w}_2^* K_2\mathbf{w}_2]/2$$

and find its derivative along trajectories of system (4.5). Substituting $\mathbf{M}$ and $\mathbf{w}_2$ into this function from Eqs (4.2) and (4.4), we obtain

$$\dot{V} = -\mathbf{w}_2^* K_2 K_3 \mathbf{w}_2$$

Let us investigate the set $S = \{\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2, \hat{\mathbf{q}}_2\colon \dot{V} = 0\}$. It follows from the condition $\mathbf{w}_2 = 0$ and from Eq. (4.3) that $\mathbf{q}_2 = \mathbf{q}_{2c} = \text{const}$. The second equation of (4.1) implies that $-k(\mathbf{q}_1 - \mathbf{q}_{2c}) = -K_0(\mathbf{q}_{2c} - \mathbf{q}_{2p})$. Hence it follows that $\mathbf{q}_1 = \mathbf{q}_{1c} = \text{const}$. It then follows from the first equation of (4.1) that $\mathbf{q}_{1c} = \mathbf{q}_{2c}$. Considering the second equation of (4.1) and (4.2) once more, we have $\mathbf{q}_{2c} = \mathbf{q}_{2p}$, while the equality $\mathbf{q}_{1c} = \mathbf{q}_{2c}$ yields $\mathbf{q}_{1c} = \mathbf{q}_{2p}$. Thus, the set S consists of the single point ($\mathbf{q}_{1p}$, 0, $\mathbf{q}_{2p}$, 0, $\mathbf{q}_{2p}$). By the Barbashin–Krasovskii theorem, this equilibrium position is asymptotically stable on the whole.

5. ASYMPTOTIC STABILIZATION OF ROTATION OF A RIGID BODY WITH A FIXED POINT

Equations of dynamics. Suppose a rigid body has a fixed point O which coincides with its centre of mass. Denote the principal central axes of inertia of the body by $Oxyz$. Suppose the mutually perpendicular unit vectors $\mathbf{s} = (s_x, s_y, s_z)'$, $\mathbf{s}_2$, $\mathbf{s}_3$ are stationary in the inertial system of coordinates, their components vary with time, while the mutually perpendicular unit vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are fixed relative to the rigid body and their components are fixed in time. The components of the vectors written from now on are their projections onto the principal axes of inertia of the rigid body.

The Euler equations for the dynamics of the rotating body may be written in the form

$$\Theta\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \Theta\boldsymbol{\omega} = \mathbf{M} \quad (\boldsymbol{\omega} = (p,q,r)', \quad \Theta = \operatorname{diag}\{A,B,C\}) \tag{5.1}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body and $A, B, C > 0$ are its principal central moments of inertia.

The kinematics of the body are described by Poisson's equations

$$\dot{\mathbf{s}} = \mathbf{s} \times \boldsymbol{\omega} \tag{5.2}$$

$$\dot{\mathbf{s}}_i = \mathbf{s}_i \times \boldsymbol{\omega}, \quad i = 2, 3 \tag{5.3}$$

Stabilization of a pair of unit vectors attached to the body. It is required to find a control law under which the ordered pair of vectors $\mathbf{s}_2$ and $\mathbf{s}_3$ will tend to the ordered pair $\mathbf{r}_2$ and $\mathbf{r}_3$ and the angular velocity $\boldsymbol{\omega}$ will tend to zero.

To solve this problem, the following control torques have been proposed in particular [8, 9]

$$\mathbf{M} = -\boldsymbol{\omega} + k\mathbf{r} \times \mathbf{s} + k_2\mathbf{r}_2 \times \mathbf{s}_2 + k_3\mathbf{r}_3 \times \mathbf{s}_3$$

where k_1, k_2 and k_3 are pairwise different positive numbers.

Below we will propose an analogous control torque using measurement of the components of only the pair of vectors $\mathbf{s}_2$ and $\mathbf{s}_3$.

Consider the control vector

$$\mathbf{M} = -\Gamma\boldsymbol{\omega} + k_2\mathbf{r}_2 \times \mathbf{s}_2 + k_3\mathbf{r}_3 \times \mathbf{s}_3 \tag{5.4}$$

where $\Gamma = \text{diag}\,\{\alpha, \beta, \gamma\}$, $\alpha, \beta, \gamma > 0$; $k_2, k_3 > 0$, $k_2 \neq k_3$.

Theorem 5.1. The closed-loop system (5.1), (5.3), (5.4) has an asymptotically stable equilibrium position $\mathbf{s}_2 = \mathbf{r}_2$, $\mathbf{s}_3 = \mathbf{r}_3$, $\boldsymbol{\omega} = 0$ and unstable equilibrium positions $\mathbf{s}_2 = -\mathbf{r}_2$, $\mathbf{s}_3 = -\mathbf{r}_3$, $\boldsymbol{\omega} = 0$; $\mathbf{s}_2 = -\mathbf{r}_2$, $\mathbf{s}_3 = \mathbf{r}_3$, $\boldsymbol{\omega} = 0$; $\mathbf{s}_2 = \mathbf{r}_2$, $\mathbf{s}_3 = -\mathbf{r}_3$, $\boldsymbol{\omega} = 0$.

Proof. Consider the Lyapunov function

$$V = \boldsymbol{\omega}\Theta\boldsymbol{\omega}/2 + k_2(\mathbf{s}_2 - \mathbf{r}_2)^2 + k_3(\mathbf{s}_3 - \mathbf{r}_3)^2 \tag{5.5}$$

whose derivative along trajectories of system (5.1), (5.3), (5.4) is

$$\dot{V} = -\boldsymbol{\omega}\Gamma\boldsymbol{\omega} \tag{5.6}$$

Let us investigate the set $\{s_2, s_3, \boldsymbol{\omega}: \dot{V} = 0\}$. Taking the equations of dynamics and the control law into consideration, we obtain

$$0 = k_2\mathbf{r}_2 \times \mathbf{s}_2 + k_3\mathbf{r}_3 \times \mathbf{s}_3 \tag{5.7}$$

Lemma. Let $\mathbf{s}_2$ and $\mathbf{s}_3$ be two mutually perpendicular unit vectors, let $\mathbf{r}_2$ and $\mathbf{r}_3$ be another pair of mutually perpendicular unit vectors, and let $\mathbf{k}_2$ and $\mathbf{k}_3$ be different positive numbers. Then it follows from (5.7) that each vector $\mathbf{s}_2$ and $\mathbf{s}_3$ is collinear with the corresponding vector $\mathbf{r}_2$ and $\mathbf{r}_3$.

Proof. Suppose the contrary. It follows from (5.7) that the vectors $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{r}_2$, $\mathbf{r}_3$ are coplanar. There are two possible situations: the pair $\mathbf{r}_2$ and $\mathbf{r}_3$ has the same orientation as $\mathbf{s}_2$ and $\mathbf{s}_3$ in the plane, or the two pairs have opposite orientations. In the first case, the non-zero angle between $\mathbf{r}_2$ and $\mathbf{s}_2$ is equal to the angle between $\mathbf{r}_3$ and $\mathbf{s}_3$ and, since k_2 and k_3 are positive, equality (5.7) is impossible. But if the pairs have opposite orientations, the angle between $\mathbf{r}_2$ and $\mathbf{s}_2$ is the complement with respect to 180° of the angle between $\mathbf{r}_3$ and $\mathbf{s}_3$, the vectors $k_2\mathbf{r}_2 \times \mathbf{s}_2$ and $k_3\mathbf{r}_3 \times \mathbf{s}_3$ have opposite directions, and equality (5.7) is impossible because the numbers k_2 and k_3 are different.

Applying Barbashin's theorem [10, p. 25] and the lemma, we deduce that the equilibrium position $\mathbf{s}_2 = \mathbf{r}_2$, $\mathbf{s}_3 = \mathbf{r}_3$, $\boldsymbol{\omega} = 0$ is asymptotically stable.

We will now prove that the equilibrium position $\mathbf{s}_2 = -\mathbf{r}_2$, $\mathbf{s}_3 = -\mathbf{r}_3$, $\boldsymbol{\omega} = 0$ is unstable. Consider a Lyapunov function differing from (5.5) in that the coefficients k_2 and k_3 have different signs. Its derivative along trajectories of system (5.1), (5.3), (5.4) is identical with (5.6). By Krasovskii's instability theorem [2], this equilibrium position is unstable.

The instability of the remaining equilibrium positions is proved in a similar fashion.

Stabilization of a pair of unit vectors attached to the body without measurement of its velocity vector. We consider a control vector similar to (5.4)

$$\mathbf{M} = \Sigma_2^3(\mathbf{s}_i \times \Gamma_i\mathbf{w}_i + k_i\mathbf{r}_i \times \mathbf{s}_i) \tag{5.8}$$

where Γ_i are diagonal matrices of gains with positive diagonal elements, $k_i > 0$ are also gains, $\mathbf{w}_i = \mathbf{s}_i - \hat{\mathbf{s}}_i$ and $\hat{\mathbf{s}}_i$ is an estimate of $\mathbf{s}_i$ ($i = 2, 3$).

We also consider the differential vector equations of observation

$$\dot{\hat{\mathbf{s}}}_i = \Delta_i(\mathbf{s}_i - \hat{\mathbf{s}}_i),\ i = 2, 3 \tag{5.9}$$

where Δ_i are diagonal matrices with positive diagonal elements. It is obvious that Eq. (3.2) are equivalent to the equations.

$$\dot{\mathbf{w}}_i = -\Delta_i\mathbf{w}_i + \dot{\mathbf{s}}_i \tag{5.10}$$

Theorem 5.2. The closed-loop system (5.1), (5.3), (5.8), (5.9) has an asymptotically stable equilibrium position

$$\mathbf{s}_2 = \mathbf{r}_2, \quad \mathbf{s}_3 = \mathbf{r}_3, \quad \boldsymbol{\omega} = 0, \quad \mathbf{w}_2 = 0, \quad \mathbf{w}_3 = 0 \tag{5.11}$$

and unstable equilibrium positions

$$\mathbf{s}_2 = -\mathbf{r}_2, \quad \mathbf{s}_3 = -\mathbf{r}_3, \quad \boldsymbol{\omega} = 0, \quad \mathbf{w}_2 = 0, \quad \mathbf{w}_3 = 0 \tag{5.12}$$

$$\mathbf{s}_2 = -\mathbf{r}_2, \quad \mathbf{s}_3 = \mathbf{r}_3, \quad \boldsymbol{\omega} = 0, \quad \mathbf{w}_2 = 0, \quad \mathbf{w}_3 = 0 \tag{5.13}$$

$$\mathbf{s}_2 = \mathbf{r}_2, \quad \mathbf{s}_3 = -\mathbf{r}_3, \quad \boldsymbol{\omega} = 0, \quad \mathbf{w}_2 = 0, \quad \mathbf{w}_3 = 0 \tag{5.14}$$

Proof. Consider the Lyapunov function

$$V = \boldsymbol{\omega}\Theta\boldsymbol{\omega}/2 + \Sigma_2^3[k_i(\mathbf{s}_i - \mathbf{r}_i)^2 + \mathbf{w}_i\Gamma_i\mathbf{w}_i]/2 \tag{5.15}$$

whose derivative along trajectories of system (5.1), (5.3), (5.4), (5.9) is

$$\dot{V} = -\Sigma_2^3\mathbf{w}_i\Gamma_i\Delta_i\mathbf{w}_i \tag{5.16}$$

Let us analyse the set $\{\mathbf{s}_2, \mathbf{s}_3, \boldsymbol{\omega}, \mathbf{w}_2, \mathbf{w}_3: \dot{V} = 0\}$. It follows from Eq. (5.2) and the condition $\mathbf{w}_2 = \mathbf{w}_3 = 0$ that $\mathbf{s}_i = \mathbf{c}_i$ $(i = 2, 3)$, where $\mathbf{c}_i$ are certain constant unit vectors. Using Eqs (5.3), we obtain $\boldsymbol{\omega}\|\mathbf{c}_2$ and $\boldsymbol{\omega}\|\mathbf{c}_3$. Since $\mathbf{c}_2$ and $\mathbf{c}_3$ are orthogonal, it follows that $\boldsymbol{\omega} = 0$. Taking the Euler equations and the control law (5.8) into consideration, we obtain (5.7). Application of the lemma and Barbashin's theorem yields the asymptotic stability of the equilibrium position (5.11).

We will now prove that the equilibrium position (5.12) is unstable. To that end, we consider a function differing from (5.15) in having a minus sign before the sum and a plus sign before $\mathbf{r}_i$. Its derivative along trajectories of system (5.1), (5.3), (5.8), (5.9) is identical with (5.11). Applying Krasovskii's instability theorem, we infer that the equilibrium position (5.12) is unstable.

The instability of the remaining equilibrium positions is proved in similar fashion.

Stabilization of the permanent rotation of a rigid body. An important particular motion of an uncontrolled rigid body is permanent rotation at a prescribed velocity $p = p_p$ about the central axis of inertia. Corresponding to this motion is the following particular solution of Eqs (5.1) and (5.2)

$$p = p_p, \quad q = r = 0, \quad s_x = 1, \quad s_y = s_z = 0 \tag{5.17}$$

Consider the following control torques

$$M_x = -\alpha(p - p_p), \; M_y = -\beta q - ks_z, \; M_z = -\gamma r + ks_y \quad (\alpha, \beta, \gamma, k > 0) \tag{5.18}$$

Theorem 5.3. The point (5.17) is an equilibrium position of the closed-loop system (5.1), (5.2), (5.18); it is asymptotically stable provided that

$$\beta\gamma > p_p^2(B - C)^2/4 \tag{5.19}$$

The system has an unstable equilibrium position obtained from (5.17) by replacing $s_x = 1$ by $s_x = -1$.

Proof. Consider the Lyapunov function

$$V = (A(p - p_p)^2 + Bq^2 + Cr^2)/2 + k((s_x - 1)^2 + s_y^2 + s_z^2)/2$$

Its derivative along trajectories of system (5.1), (5.2), (5.17) is

$$\dot{V} = -\alpha(p - p_p)^2 - \beta q^2 - \gamma r^2 - p_p(B - C)qr$$

By Sylvester's criterion, this form is negative definite in the variables $p - p_p, q, r$, provided that (5.19) holds. Let us investigate the set $\{p - p_p, q, r, s: V = 0\}$. It follows from the condition $p - p_p = 0, q = r = 0$ and from Eqs (5.1) and (5.18) that $0 = -ks_z$, $0 = -ks_y$. Hence $s_y = s_z = 0$ and so $s_x = \pm 1$ by the normalization condition for the vector **s**. Applying Barbashin's asymptotic stability theorem, we conclude that equilibrium position (5.17) is asymptotically stable.

The instability of the other equilibrium position may be verified by analysing the eigenvalues of the linear approximation or by using Krasovskii's instability theorem.

Stabilization by two control torques with a measurement of the velocities. Let us assume that the body is controlled by two torques applied along its axes of inertia, that is, $M_x = 0$ in Eq. (1.1). It turns out that the body is then asymptotically stabilizable to permanent rotation at a certain velocity $p = c$.

Consider the control torques

$$M_y = -\beta q - ks_z, \quad M_z = -\gamma r + ks_y \quad (\beta, \gamma, k > 0) \tag{5.20}$$

Theorem 5.4. The closed-loop system (5.1), (5.2), (5.20) has an asymptotically stable family of non-isolated equilibrium positions $p = c, q = r = 0, s_x = 1, s_y = s_z = 0$ and an unstable family of non-isolated equilibrium positions $p = c, q = r = 0, s_x = -1, s_y = s_z = 0$ (where c is a parameter).

Proof. Consider the Lyapunov function

$$V = (Ap^2 + Bq^2 + Cr^2)/2 + k((s_x - 1)^2 + s_y^2 + s_z^2)/2$$

Calculating its derivative along trajectories of system (5.1), (5.2), (5.20), we obtain $\dot{V} = -\beta q^2 - \gamma r^2$.

We will investigate the set $\{s, \boldsymbol{\omega}: \dot{V} = 0\}$, using Euler's equations and the condition $q = r = 0$, which yield $A\dot{p} = 0, 0 = -ks_z, 0 = ks_y$. Consequently, $p = c$, where c is some constant and $s_y = s_z = 0$. Noting the normalization condition for **s**, we obtain $s_x = \pm 1$. We then apply La Salle's theorem [11].

The instability of the family of equilibrium positions defined by $s_x = -1$ may be established by using Krasovskii's instability theorem.

6. REMARKS

The proof that regulator (2.1) is asymptotically stable on the whole is an extension of the proof that the proportional-plus-differential regulator $\mathbf{M} = -K_2(\mathbf{q} - \mathbf{q}_p) - K_2\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})$ for natural mechanical systems is asymptotically stable [12]. It has been proved [13] that a proportional-plus-integral-differential regulator guarantees asymptotic stability on the whole of a natural mechanical system. The proof that regulator (4.2) is asymptotically stable is an extension of the asymptotic stability proof for a proportional-plus-differential regulator in an elastic manipulator [14].

It has been shown [15] that a natural mechanical system may be asymptotically stabilized on the whole using bounded controls (of the type $M_i = \operatorname{arctg}(-k_{0i}(q_i - q_{pi}) - k_{1i}\dot{q}_i) + g_i(\mathbf{q})$).

Various papers [16–19] have proposed non-linear velocity observers, which are complicated to implement.

The stabilization methods of [20–24], which use non-linear velocity observers and non-linear control laws, require a large amount of real-time computations.

The method proposed above has the advantage that the differential equations of observation are of order n, where n is the number of degrees of freedom, and not $2n$, as in [20–24]. In addition, the regulator and observer are linear and can be implemented in real-time operation, without the use of computers. As the observation equations are of order n, our results are comparable with earlier results [25] for single-input linear systems.

The control methods of [20–24] also require a knowledge of the exact form of the object's equations of dynamics, in particular, of the system parameters. Control scheme (3.2), (3.7) does not require this, while control scheme (2.1), (2.2) requires a knowledge of the potential forces $g(q)$ only.

The merit of [19, 20] is that they present estimates for the gain and the stability region.

The control laws proposed here for a rigid body do not require a knowledge of the inertia parameters A, B and C.

Since the control laws are synthesized through the use of Lyapunov functions, the attraction domains are easy to investigate.

Asymptotic stabilization of a rigid body (on the assumption that the kinematics are described by the Euler–Krylov angles) was investigated in [26].

The problem of the asymptotic stabilization of permanent rotation using only velocities was considered in [27]. There is an inaccuracy in Theorem 3.1 of [27]. The system investigated there in fact has the property of local asymptotic stability, rather than stability in the large, since the closed-loop system has a second unstable equilibrium position ($\gamma_3 = -1$ in the notation of [27]). The restrictions considered above on the gains β and γ in Theorem 4.1 are exactly the conditions of [27].

It is well known [28] that permanent rotations of an uncontrollable rigid body about the major and minor axes of the inertia ellipsoid are stable with respect to velocities, while rotations about the middle axis are unstable.

It has been shown [29] that the Euler equations can be asymptotically stabilized in all three velocities by two control torques.

The stabilization of the Euler equations of a rigid body one and two control torques was considered in [30, p. 143].

The control torques proposed in Section 5 differ from the ones previously investigated [31, 32] in their greater simplicity; moreover, the explicitly specified Lyapunov formula makes it possible to investigate the attraction domain.

An angular velocity observer for a rigid body, using only a measurement of the position coordinates, was constructed in [33], but the problem of stabilizing the rotation of a rigid body was not considered.

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